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Control of Nonlinear Systems by Fuzzy Observer-Controller with Unmeasurable Premise Variables

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Abstract—In this paper, we investigate the stability of Takagi-Sugeno (T-S) fuzzy-model-based (FMB) observer-control system. Premise membership functions depending on unmeasurable premise variables are considered to enhance the flexibility and applicability of the fuzzy observer-controller. The fuzzy observer is designed to estimate the system states and the estimated states are employed for state-feedback control of nonlinear systems. Convex stability conditions are obtained through matrix decoupling technique so that a solution can be searched using convex programming techniques. The proposed analysis is able to reduce the number of predefined scalars by adequately choosing the augmented vector. Nonetheless, due to the mismatched premise membership functions between the fuzzy model and fuzzy observer-controller, it complicates the stability analysis which potentially leads to conservative stability conditions. To alleviate the problem, the stability conditions are relaxed by membership-function-dependent approach which takes the information of membership functions into consideration in the stability analysis. Simulation examples are provided to demonstrate the feasibility and relaxation of proposed FMB observer-control scheme.

I. INTRODUCTION

Stability of nonlinear systems is difficult to be analyzed due to the system complexity such as nonlinearity, unmeasurable system states and etc. Fuzzy-model-based (FMB) control strategy is a systematic and efficient approach to support analysis and control design for nonlinear systems by representing the nonlinear system with a Takagi-Sugeno (T-S) fuzzy model [1] and closing the feedback loop with a fuzzy controller. The sector nonlinearity technique [2], [3] is employed to handle the nonlinearity and establish the T-S fuzzy model and polynomial fuzzy model [4] to precisely describe the nonlinear systems. In this way, the nonlinear systems are separated to several linear subsystems which are smoothly combined by membership functions. As a result, linear control techniques can be applied such as state-feedback control. Based on the T-S fuzzy model, stability analysis can be carried out through Lyapunov stability theory [5]. In order to numerically solve stability conditions, linear matrix inequalities (LMIs) [6] and sum of squares (SOS) [7] are employed to describe the stability conditions for T-S and polynomial fuzzy models, respectively.

Therefore, the stability can be guaranteed and the feedback gains can be simultaneously obtained if there exists a feasible solution to the stability conditions.

Although FMB control scheme can be applied to nonlinear systems, the conservativeness of stability conditions needs to be reduced [8] such that it can be applied to a wider class of nonlinear systems. The first method of reducing the conservativeness is considering the permutations of membership functions in the fuzzy summations [9], [10], which can be handled generally by Pólya's theory in [11]. The second approach is exploiting different Lyapunov function candidates such as quadratic Lyapunov function [5], piecewise linear Lyapunov function [12], switching Lyapunov function [13], [14], fuzzy Lyapunov function [15], [16] and polynomial Lyapunov function [14], [17]. The third method is obtaining membership-function-dependent stability conditions. By bringing the information of membership functions into stability analysis, the stability conditions will depend on particular shapes of membership functions rather than any shapes. This approach includes polynomial constraints [18], symbolic variables [3], [19], [20], approximated membership functions [21], [22] and others [16]. During the relaxation process, slack matrices are added to stability conditions through S-procedure [23], which brings more freedom for satisfying the conditions.

With relaxed stability conditions being extensively investigated, FMB control strategy is applied to various control problems, for instance, output feedback [24], uncertainty [25] and sampled-data system [26], [27]. As one of the output feedback control schemes, fuzzy observer was proposed to estimate system states according to the system outputs [6]. If measurable premise variables are used in membership function, separation principle [28] can be employed to independently design the fuzzy controller and the fuzzy observer. However, the assumption of measurable premise variables are only valid for a limited class of nonlinear systems. To increase the applicability of the fuzzy observer, membership functions depending on unmeasurable states were considered in [29], where a two-step procedure was required due to the non-

convex stability conditions [29]. Therefore, several techniques were proposed to transform the non-convex stability conditions to convex ones, for example, completing squares [30], matrix decoupling [31], Finsler's lemma [32] and descriptor representation [33]. In [31], although the conditions are convex, there are a number of scalars to be predefined by users or other numerical methods such as the genetic algorithm. Furthermore, the conditions are conservative resulting from approximations of non-convex terms.

Since both relaxation and membership functions in unmeasurable premise variables are important for widening the applicability of FMB observer-control scheme, it motivates us to investigate relaxed stability conditions for T-S FMB observer-control systems with unmeasurable premise variables in this paper. To achieve convex stability conditions, the matrix decoupling technique [31] is employed. Different from [31], the augmented vector is adequately chosen such that no more approximated transformation (such as completing squares) is required before applying the decoupling technique. As a result, the number of predefined scalars can be reduced. However, the stability conditions are still conservative without applying any relaxation techniques. Consequently, membership-function-dependent approach is applied to bring the upper bounds of membership functions into stability conditions through slack matrices. For the proposed fuzzy observer-controller, only two scalars are required to be predefined by users and the fuzzy observer cannot be replaced by the linear observer.

II. PRELIMINARY

A. Notation

The following notation is employed throughout this paper. The expressions of $\mathbf{M} > 0$, $\mathbf{M} \geq 0$, $\mathbf{M} < 0$, and $\mathbf{M} \leq 0$ denote the positive, semi-positive, negative, and semi-negative definite matrices \mathbf{M} , respectively. The symbol “*” in a matrix represents the transposed entry in the corresponding position.

B. T-S Fuzzy Model

The i^{th} rule of the T-S fuzzy model is [1]:

$$\begin{aligned} \text{Rule } i: & \text{ IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i, \\ & \text{ THEN } \dot{\mathbf{x}}(t) = \mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t), \\ & \mathbf{y}(t) = \mathbf{C} \mathbf{x}(t), \end{aligned}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T$ is the state vector, and n is the dimension of the nonlinear system; $f_\eta(\mathbf{x}(t))$ is the premise variable corresponding to its fuzzy term M_η^i in rule i , $\eta = 1, 2, \dots, \Psi$, and Ψ is a positive integer; $\mathbf{A}_i \in \mathbb{R}^{n \times n}$ and $\mathbf{B}_i \in \mathbb{R}^{n \times m}$ are the known system and input matrices, respectively; $\mathbf{u}(t) \in \mathbb{R}^m$ is the control input vector; $\mathbf{y}(t) \in \mathbb{R}^l$ is the output vector; $\mathbf{C} \in \mathbb{R}^{l \times n}$ is the output matrix. The dynamics of the nonlinear system is given by

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \sum_{i=1}^p w_i(\mathbf{x}(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)), \\ \mathbf{y}(t) &= \mathbf{C} \mathbf{x}(t), \end{aligned} \quad (1)$$

where p is the number of fuzzy rules; $w_i(\mathbf{x}(t))$ is the normalized grade of membership, $w_i(\mathbf{x}(t)) = \frac{\prod_{\eta=1}^{\Psi} \mu_{M_\eta^i}(f_\eta(\mathbf{x}(t)))}{\sum_{k=1}^p \prod_{\eta=1}^{\Psi} \mu_{M_\eta^k}(f_\eta(\mathbf{x}(t)))}$, $w_i(\mathbf{x}(t)) \geq 0$, $i = 1, 2, \dots, p$, and $\sum_{i=1}^p w_i(\mathbf{x}(t)) = 1$; $\mu_{M_\eta^i}(f_\eta(\mathbf{x}(t)))$, $\eta = 1, 2, \dots, \Psi$, are grades of membership corresponding to the fuzzy term M_η^i .

C. T-S Fuzzy Observer

For brevity, time t is dropped for variables from now. Considering the premise variable $f_\eta(\mathbf{x})$ depending on unmeasurable states, we apply the T-S fuzzy observer with its i^{th} rule described as follows:

$$\begin{aligned} \text{Rule } i: & \text{ IF } f_1(\check{\mathbf{x}}) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\check{\mathbf{x}}) \text{ is } M_\Psi^i, \\ & \text{ THEN } \dot{\check{\mathbf{x}}} = \mathbf{A}_i \check{\mathbf{x}} + \mathbf{B}_i \mathbf{u} + \mathbf{L}_i (\mathbf{y} - \check{\mathbf{y}}), \\ & \check{\mathbf{y}} = \mathbf{C} \check{\mathbf{x}}, \end{aligned}$$

where $\check{\mathbf{x}} \in \mathbb{R}^n$ is the estimated state \mathbf{x} ; $\check{\mathbf{y}} \in \mathbb{R}^l$ is the estimated output \mathbf{y} ; $\mathbf{L}_i \in \mathbb{R}^{n \times l}$ is the observer gain. The T-S fuzzy observer is given by

$$\begin{aligned} \dot{\check{\mathbf{x}}} &= \sum_{i=1}^p w_i(\check{\mathbf{x}}) (\mathbf{A}_i \check{\mathbf{x}} + \mathbf{B}_i \mathbf{u} + \mathbf{L}_i (\mathbf{y} - \check{\mathbf{y}})), \\ \check{\mathbf{y}} &= \mathbf{C} \check{\mathbf{x}}. \end{aligned} \quad (2)$$

D. T-S Fuzzy Controller

Using the parallel distributed compensation (PDC) approach [5], the i^{th} rule of the T-S fuzzy controller is:

$$\begin{aligned} \text{Rule } i: & \text{ IF } f_1(\check{\mathbf{x}}) \text{ is } M_1^i \text{ AND } \cdots \text{ AND } f_\Psi(\check{\mathbf{x}}) \text{ is } M_\Psi^i, \\ & \text{ THEN } \mathbf{u} = \mathbf{G}_i \check{\mathbf{x}}, \end{aligned}$$

where $\mathbf{G}_i \in \mathbb{R}^{m \times n}$ is the controller gain. The T-S fuzzy controller is given by

$$\mathbf{u} = \sum_{i=1}^p w_i(\check{\mathbf{x}}) \mathbf{G}_i \check{\mathbf{x}}. \quad (3)$$

III. STABILITY ANALYSIS

In this section, stability analysis is conducted for T-S FMB observer-control systems. The closed-loop systems are provided first. Then based on the augmented systems and the Lyapunov stability theory, we derive the convex stability conditions by matrix decoupling technique. Finally, the membership-function-dependent approach is applied to relax the stability conditions.

The estimation error is defined as $\mathbf{e} = \mathbf{x} - \check{\mathbf{x}}$, and then we have the closed-loop systems:

$$\dot{\mathbf{x}} = \sum_{i=1}^p \sum_{k=1}^p w_i(\mathbf{x}) w_k(\check{\mathbf{x}}) ((\mathbf{A}_i + \mathbf{B}_i \mathbf{G}_k) \check{\mathbf{x}} + \mathbf{A}_i \mathbf{e}), \quad (4)$$

$$\dot{\check{\mathbf{x}}} = \sum_{j=1}^p \sum_{k=1}^p w_j(\check{\mathbf{x}}) w_k(\check{\mathbf{x}}) ((\mathbf{A}_j + \mathbf{B}_j \mathbf{G}_k) \check{\mathbf{x}} + \mathbf{L}_j \mathbf{C} \mathbf{e}), \quad (5)$$

$$\begin{aligned} \dot{\mathbf{e}} &= \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p w_i(\mathbf{x}) w_j(\check{\mathbf{x}}) w_k(\check{\mathbf{x}}) ((\mathbf{A}_i - \mathbf{A}_j \\ &+ (\mathbf{B}_i - \mathbf{B}_j) \mathbf{G}_k) \check{\mathbf{x}} + (\mathbf{A}_i - \mathbf{L}_j \mathbf{C}) \mathbf{e}). \end{aligned} \quad (6)$$

Theorem 1: The augmented T-S FMB observer-control system (formed by (5) and (6)) is guaranteed to be asymptotically stable if there exist matrices $\mathbf{X} \in \mathbb{R}^{n \times n}$, $\mathbf{Y} \in \mathbb{R}^{m \times n}$, $\mathbf{N}_k \in \mathbb{R}^{m \times n}$, $\mathbf{M}_j \in \mathbb{R}^{n \times l}$, $\mathbf{R}_{ijk} = \mathbf{R}_{ikj} \in \mathbb{R}^{3n \times 3n}$, $\mathbf{S}_{ij} \in \mathbb{R}^{3n \times 3n}$, $i, j, k = 1, 2, \dots, p$, and predefined scalars $\alpha_1 > 0$, $\alpha_2 > 0$ such that the following LMI-based conditions are satisfied:

$$\mathbf{X} > 0; \quad (7)$$

$$\mathbf{Y} > 0; \quad (8)$$

$$\mathbf{R}_{ijk} \geq 0 \quad \forall i \text{ and } j \leq k; \quad (9)$$

$$\mathbf{S}_{ij} \geq 0 \quad \forall i, j; \quad (10)$$

$$\Phi_{ijk} + \Phi_{ikj} - 2\mathbf{R}_{ijk} + 2 \sum_{l=1}^p \sum_{m=1}^p \sum_{n=1}^p \gamma_{lmn} \mathbf{R}_{lmn} < 0 \quad \forall i \text{ and } j \leq k; \quad (11)$$

$$\Theta_{ij} - \mathbf{S}_{ij} + \sum_{l=1}^p \sum_{m=1}^p \gamma_{lm} \mathbf{S}_{lm} < 0 \quad \forall i, j; \quad (12)$$

where

$$\Phi_{ijk} = \begin{bmatrix} \hat{\Xi}_{jk}^{(11)} + \hat{\Xi}_{jk}^{(11)T} & \hat{\Xi}_{ijk}^{(21)T} & \mathbf{I} \\ * & -\alpha_2 \mathbf{I} & \mathbf{0} \\ * & * & -\frac{1}{\alpha_1} \mathbf{Y} \end{bmatrix}, \quad (13)$$

$$\Theta_{ij} = \begin{bmatrix} -\alpha_1 \mathbf{Y} & \tilde{\Xi}_j^{(12)} & \mathbf{0} \\ * & \tilde{\Xi}_{ij}^{(22)} + \tilde{\Xi}_{ij}^{(22)T} & \mathbf{Y} \\ * & * & -\frac{1}{\alpha_2} \mathbf{I} \end{bmatrix}, \quad (14)$$

$$\hat{\Xi}_{jk}^{(11)} = \mathbf{A}_j \mathbf{X} + \mathbf{B}_j \mathbf{N}_k, \quad (15)$$

$$\hat{\Xi}_{ijk}^{(21)} = (\mathbf{A}_i - \mathbf{A}_j) \mathbf{X} + (\mathbf{B}_i - \mathbf{B}_j) \mathbf{N}_k, \quad (16)$$

$$\tilde{\Xi}_j^{(12)} = \mathbf{M}_j \mathbf{C}, \quad (17)$$

$$\tilde{\Xi}_{ij}^{(22)} = \mathbf{Y} \mathbf{A}_i - \mathbf{M}_j \mathbf{C}, \quad (18)$$

γ_{ijk} and γ_{ij} are the upper bounds of membership functions $w_i(\mathbf{x})w_j(\check{\mathbf{x}})w_k(\check{\mathbf{x}})$ and $w_i(\mathbf{x})w_j(\check{\mathbf{x}})$, respectively; and the controller and observer gains are given by $\mathbf{G}_k = \mathbf{N}_k \mathbf{X}^{-1}$ and $\mathbf{L}_j = \mathbf{Y}^{-1} \mathbf{M}_j$, respectively.

Proof: Defining the augmented vector $\mathbf{z} = [\check{\mathbf{x}}^T \quad \mathbf{e}^T]^T$ and denote $w_i(\mathbf{x})w_j(\check{\mathbf{x}})w_k(\check{\mathbf{x}})$ as h_{ijk} and $w_i(\mathbf{x})w_j(\check{\mathbf{x}})$ as h_{ij} , the augmented T-S FMB observer-control system is written as

$$\dot{\mathbf{z}} = \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \Xi_{ijk} \mathbf{z}, \quad (19)$$

where

$$\Xi_{ijk} = \begin{bmatrix} \Xi_{jk}^{(11)} & \Xi_{jk}^{(12)} \\ \Xi_{ijk}^{(21)} & \Xi_{ij}^{(22)} \end{bmatrix}, \quad (20)$$

$$\Xi_{jk}^{(11)} = \mathbf{A}_j + \mathbf{B}_j \mathbf{G}_k, \quad (21)$$

$$\Xi_{ijk}^{(21)} = \mathbf{A}_i - \mathbf{A}_j + (\mathbf{B}_i - \mathbf{B}_j) \mathbf{G}_k, \quad (22)$$

$$\Xi_j^{(12)} = \mathbf{L}_j \mathbf{C}, \quad (23)$$

$$\Xi_{ij}^{(22)} = \mathbf{A}_i - \mathbf{L}_j \mathbf{C}. \quad (24)$$

Remark 1: In this paper, we employ the augmented vector $\mathbf{z} = [\check{\mathbf{x}}^T \quad \mathbf{e}^T]^T$ rather than $\mathbf{z} = [\mathbf{x}^T \quad \mathbf{e}^T]^T$ in [31]. This is in favor of the following derivation by directly separating the controller-related decision matrices from the observer-related decision matrices. In this way, the matrix decoupling technique [31] can be applied without any other approximated transformation. As a result, the number of predefined scalars can be reduced.

The following Lyapunov function candidate is employed to investigate the stability of the augmented T-S FMB observer-control system (19):

$$V(\mathbf{z}) = \mathbf{z}^T \mathbf{P} \mathbf{z}, \quad (25)$$

where $\mathbf{P} = \begin{bmatrix} \mathbf{X}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} \end{bmatrix}$, $\mathbf{X} > 0$, $\mathbf{Y} > 0$, and thus $\mathbf{P} > 0$. The time derivative of the Lyapunov function is given as follows:

$$\dot{V}(\mathbf{z}) = \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \mathbf{z}^T (\mathbf{P} \Xi_{ijk} + \Xi_{ijk}^T \mathbf{P}) \mathbf{z}. \quad (26)$$

Therefore, $\dot{V}(\mathbf{z}) < 0$ holds if

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} (\mathbf{P} \Xi_{ijk} + \Xi_{ijk}^T \mathbf{P}) < 0. \quad (27)$$

Remark 2: The augmented T-S FMB observer-control system (19) is guaranteed to be asymptotically stable if $V(\mathbf{z}) > 0$ and $\dot{V}(\mathbf{z}) < 0$ excluding $\mathbf{x} = \mathbf{0}$. To ensure $\dot{V}(\mathbf{z}) < 0$, in the following, the congruence transformation is employed first, which is in favor of matrix decoupling.

Performing congruence transformation to (27) by pre-multiplying and post-multiplying $\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}^{-1} \end{bmatrix}$ to both sides and denoting $\mathbf{N}_k = \mathbf{G}_k \mathbf{X}$, we have

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} (\hat{\Xi}_{ijk} + \hat{\Xi}_{ijk}^T) < 0, \quad (28)$$

where

$$\hat{\Xi}_{ijk} = \begin{bmatrix} \hat{\Xi}_{jk}^{(11)} & \hat{\Xi}_j^{(12)} \\ \hat{\Xi}_{ijk}^{(21)} & \hat{\Xi}_{ij}^{(22)} \end{bmatrix}, \quad (29)$$

$$\hat{\Xi}_j^{(12)} = \mathbf{L}_j \mathbf{C} \mathbf{Y}^{-1}, \quad (30)$$

$$\hat{\Xi}_{ij}^{(22)} = \mathbf{A}_i \mathbf{Y}^{-1} - \mathbf{L}_j \mathbf{C} \mathbf{Y}^{-1}, \quad (31)$$

$\hat{\Xi}_{jk}^{(11)}$ and $\hat{\Xi}_{ijk}^{(21)}$ are defined in (15) and (16), respectively.

Remark 3: The stability contritions (28) are non-convex, which cannot be solved by current convex programming toolboxes. Note that the controller-related decision matrices \mathbf{X} and \mathbf{G}_k are separated from the observer-related decision matrices \mathbf{Y} and \mathbf{L}_j , and only the observer-related matrices are non-convex. Therefore, the matrix decoupling technique [31] is exploited in the following such that more transformation can be enforced on the observer-related matrices without affecting controller-related matrices.

Using matrix decoupling technique [31] to further separate decision variables in order to obtain convex LMI stability conditions, we rewrite $\hat{\Xi}_{ijk} + \hat{\Xi}_{ijk}^T$ as follows:

$$\hat{\Xi}_{ijk} + \hat{\Xi}_{ijk}^T = \mathbf{\Gamma}_{ijk} + \mathbf{\Lambda}_{ij}, \quad (32)$$

where

$$\mathbf{\Gamma}_{ijk} = \begin{bmatrix} \hat{\Xi}_{jk}^{(11)} + \hat{\Xi}_{jk}^{(11)T} + \alpha_1 \mathbf{Y}^{-1} & \hat{\Xi}_{ijk}^{(21)T} \\ * & -\alpha_2 \mathbf{I} \end{bmatrix}, \quad (33)$$

$$\mathbf{\Lambda}_{ij} = \begin{bmatrix} -\alpha_1 \mathbf{Y}^{-1} & \hat{\Xi}_j^{(12)} \\ * & \hat{\Xi}_{ij}^{(22)} + \hat{\Xi}_{ij}^{(22)T} + \alpha_2 \mathbf{I} \end{bmatrix}. \quad (34)$$

Hence, $\dot{V}(\mathbf{z}) < 0$ holds if

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \mathbf{\Gamma}_{ijk} < 0, \quad (35)$$

$$\sum_{i=1}^p \sum_{j=1}^p h_{ij} \mathbf{\Lambda}_{ij} < 0. \quad (36)$$

Performing congruence transformation to (36) by pre-multiplying and post-multiplying $\text{diag}\{\mathbf{Y}, \mathbf{Y}\}$ to both sides, denoting $\mathbf{M}_j = \mathbf{Y} \mathbf{L}_j$, and then applying Schur Complement to both (35) and (36), we obtain

$$\sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \mathbf{\Phi}_{ijk} < 0, \quad (37)$$

$$\sum_{i=1}^p \sum_{j=1}^p h_{ij} \mathbf{\Theta}_{ij} < 0, \quad (38)$$

where $\mathbf{\Phi}_{ijk}$ and $\mathbf{\Theta}_{ij}$ are defined in (13) and (14), respectively.

Remark 4: Although the stability contritions (37) and (38) are convex now, they are conservative since they are membership-function-independent. Moreover, if there exists $\mathbf{M}_j \forall j$ such that (38) is satisfied, then one can let $\mathbf{M}_j = \mathbf{M}_1 \forall j$ such that (38) is also satisfied. It means that the fuzzy observer can be replaced by a linear observer, and there is no need to use a fuzzy observer. In the following, we try to relax the stability conditions by considering the information of membership functions. In this way, the advantage of fuzzy observer over linear observer can be revealed.

Defining the upper bounds of membership functions h_{ijk} and h_{ij} as γ_{ijk} and γ_{ij} , respectively, we have $\gamma_{ijk} - h_{ijk} \geq 0$ and $\gamma_{ij} - h_{ij} \geq 0$. Adding these information and slack matrices $0 \leq \mathbf{R}_{ijk} = \mathbf{R}_{ikj} \in \mathbb{R}^{3n \times 3n}$ and $0 \leq \mathbf{S}_{ij} \in \mathbb{R}^{3n \times 3n}$, we have

$$\begin{aligned} & \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \mathbf{\Phi}_{ijk} \\ & \leq \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \mathbf{\Phi}_{ijk} + \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p (\gamma_{ijk} - h_{ijk}) \mathbf{R}_{ijk} \\ & = \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \left(\mathbf{\Phi}_{ijk} - \mathbf{R}_{ijk} + \sum_{l=1}^p \sum_{m=1}^p \sum_{n=1}^p \gamma_{lmn} \mathbf{R}_{lmn} \right) \\ & = \frac{1}{2} \sum_{i=1}^p \sum_{j=1}^p \sum_{k=1}^p h_{ijk} \left(\mathbf{\Phi}_{ijk} + \mathbf{\Phi}_{ikj} - 2\mathbf{R}_{ijk} \right) \end{aligned}$$

$$+ 2 \sum_{l=1}^p \sum_{m=1}^p \sum_{n=1}^p \gamma_{lmn} \mathbf{R}_{lmn} \Big). \quad (39)$$

Similarly,

$$\begin{aligned} & \sum_{i=1}^p \sum_{j=1}^p h_{ij} \mathbf{\Theta}_{ij} \\ & \leq \sum_{i=1}^p \sum_{j=1}^p h_{ij} \left(\mathbf{\Theta}_{ij} - \mathbf{S}_{ij} + \sum_{l=1}^p \sum_{m=1}^p \gamma_{lm} \mathbf{S}_{lm} \right). \end{aligned} \quad (40)$$

Therefore, $\dot{V}(\mathbf{z}) < 0$ can be achieved by satisfying conditions (11) and (12). The proof is completed. ■

IV. SIMULATION EXAMPLES

Two simulation examples are provided to show the advantages of the proposed fuzzy observer-controller. In the first example, we compare the proposed stability conditions with those without slack matrices to demonstrate the merit of relaxation. In the second example, we compare the fuzzy observer with the linear observer to exhibit the superiority of the fuzzy observer as well as the effect of slack matrices.

A. Example 1

Consider the following T-S fuzzy model extended from [30]:

$$\begin{aligned} \mathbf{A}_1 &= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 2.5 & 0 \\ -2.3 & -1 \end{bmatrix}, \\ \mathbf{A}_3 &= \begin{bmatrix} 1.5 & -0.3 \\ 0 & -1 \end{bmatrix}, \mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \mathbf{B}_3 &= \begin{bmatrix} 1.3 \\ 0.2 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 10 & 2 \end{bmatrix}, \end{aligned}$$

where the membership functions are $w_1(x_1) = 1 - 1/(1 + e^{-(x_1+0.8)})$, $w_2(x_1) = 1 - w_1(x_1) - w_3(x_1)$, and $w_3(x_1) = 1/(1 + e^{-(x_1-0.8)})$. Defining the region of interest as $x_1 \in [-10, 10]$, we obtain the upper bounds of membership functions as $\gamma_{111} = 9.9970 \times 10^{-1}$, $\gamma_{121} = 1.6970 \times 10^{-1}$, $\gamma_{131} = 9.6106 \times 10^{-2}$, $\gamma_{211} = 3.7987 \times 10^{-1}$, $\gamma_{221} = 6.4484 \times 10^{-2}$, $\gamma_{231} = 3.6519 \times 10^{-2}$, $\gamma_{311} = 9.9970 \times 10^{-1}$, $\gamma_{321} = 1.6970 \times 10^{-1}$, $\gamma_{331} = 9.6106 \times 10^{-2}$, $\gamma_{112} = 1.6970 \times 10^{-1}$, $\gamma_{122} = 1.4435 \times 10^{-1}$, $\gamma_{132} = 1.6970 \times 10^{-1}$, $\gamma_{212} = 6.4484 \times 10^{-2}$, $\gamma_{222} = 5.4850 \times 10^{-2}$, $\gamma_{232} = 6.4484 \times 10^{-2}$, $\gamma_{312} = 1.6970 \times 10^{-1}$, $\gamma_{322} = 1.4435 \times 10^{-1}$, $\gamma_{332} = 1.6970 \times 10^{-1}$, $\gamma_{113} = 9.6106 \times 10^{-2}$, $\gamma_{123} = 1.6970 \times 10^{-1}$, $\gamma_{133} = 9.9970 \times 10^{-1}$, $\gamma_{213} = 3.6519 \times 10^{-2}$, $\gamma_{223} = 6.4484 \times 10^{-2}$, $\gamma_{233} = 3.7987 \times 10^{-1}$, $\gamma_{313} = 9.6106 \times 10^{-2}$, $\gamma_{323} = 1.6970 \times 10^{-1}$, $\gamma_{333} = 9.9970 \times 10^{-1}$, $\gamma_{11} = 9.9980 \times 10^{-1}$, $\gamma_{12} = 3.7991 \times 10^{-1}$, $\gamma_{13} = 9.9980 \times 10^{-1}$, $\gamma_{21} = 3.7991 \times 10^{-1}$, $\gamma_{22} = 1.4436 \times 10^{-1}$, $\gamma_{23} = 3.7991 \times 10^{-1}$, $\gamma_{31} = 9.9980 \times 10^{-1}$, $\gamma_{32} = 3.7991 \times 10^{-1}$, $\gamma_{33} = 9.9980 \times 10^{-1}$.

This example shows that the proposed stability conditions are more relaxed than [31] which does not include any slack matrices. Choosing $\alpha_1 = 5.7$, $\alpha_2 = 1$ and applying Theorem 1, we obtain a feasible solution. The controller gains are $\mathbf{G}_1 = [-1.3609 \times 10^1 \quad 2.8117 \times 10^{-2}]$, $\mathbf{G}_2 = [-1.5259 \times 10^1 \quad 1.3524 \times 10^{-3}]$ and $\mathbf{G}_3 = [-1.3798 \times 10^1 \quad -6.4585 \times$

$10^{-2}]$, and the observer gains are $\mathbf{L}_1 = [6.3479 \times 10^{-1} \quad -3.2689 \times 10^{-1}]^T$, $\mathbf{L}_2 = [6.3371 \times 10^{-1} \quad -3.2541 \times 10^{-1}]^T$ and $\mathbf{L}_3 = [6.3479 \times 10^{-1} \quad -3.2689 \times 10^{-1}]^T$. Choosing the initial conditions $\mathbf{x}(0) = [9 \quad 9]^T$ and $\hat{\mathbf{x}}(0) = [0 \quad 0]^T$, the responses of system and estimated states are shown in Fig. 1.

For comparison purposes, we set $\mathbf{R}_{ijk} = \mathbf{0}$ and $\mathbf{S}_{ij} = \mathbf{0} \forall i, j, k$ in Theorem 1 to investigate how the slack matrix variables influence the conservativeness of the stability conditions. While other settings are the same, no feasible solutions can be found. Consequently, the proposed stability conditions can be further relaxed than other existing results without exploiting any information of membership functions.

B. Example 2

Consider the following T-S fuzzy model:

$$\mathbf{A}_1 = \begin{bmatrix} 2.5 & 0 \\ -2.4 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 2.5 & 0 \\ -2.3 & -1 \end{bmatrix},$$

$$\mathbf{B}_1 = \mathbf{B}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{C} = [10 \quad 2],$$

where the membership functions are $w_1(x_2) = 0.5 + \frac{\arctan(x_2-3.2)}{\pi}$ and $w_2(x_2) = 1 - w_1(x_2)$. Defining the region of interest as $x_2 \in [-1.8, 1.8]$, we obtain the upper bounds $\gamma_{111} = 7.6957 \times 10^{-3}$, $\gamma_{121} = 3.1283 \times 10^{-2}$, $\gamma_{211} = 3.6530 \times 10^{-2}$, $\gamma_{221} = 1.4850 \times 10^{-1}$, $\gamma_{112} = 3.1283 \times 10^{-2}$, $\gamma_{122} = 1.7340 \times 10^{-1}$, $\gamma_{212} = 1.4850 \times 10^{-1}$, $\gamma_{222} = 8.2310 \times 10^{-1}$, $\gamma_{11} = 3.8979 \times 10^{-2}$, $\gamma_{12} = 1.8503 \times 10^{-1}$, $\gamma_{21} = 1.8503 \times 10^{-1}$, $\gamma_{22} = 8.7828 \times 10^{-1}$.

In this example, we aim to demonstrate that the proposed fuzzy observer cannot be replaced by the linear observer. Choosing $\alpha_1 = 5.0001$, $\alpha_2 = 1$ and applying Theorem 1, we obtain a feasible solution with controller gains as $\mathbf{G}_1 = [-2.2459 \times 10^3 \quad 3.9537 \times 10^{-3}]$ and $\mathbf{G}_2 = [-2.3983 \times 10^3 \quad 4.2161 \times 10^{-3}]$ and observer gains as $\mathbf{L}_1 = [5.7609 \times 10^{-1} \quad -3.8037 \times 10^{-1}]^T$ and $\mathbf{L}_2 = [5.7610 \times 10^{-1} \quad -3.8044 \times 10^{-1}]^T$. Choosing the initial conditions $\mathbf{x}(0) = [1 \quad 1.8]^T$ and $\hat{\mathbf{x}}(0) = [0 \quad 0]^T$, the corresponding time responses are shown in Fig. 2.

To design the linear observer, we let $\mathbf{M}_j = \mathbf{M} \forall j$ in Theorem 1 and keep other settings the same. However, no feasible solutions can be found. It indicates that the proposed fuzzy observer is more general than the linear observer, which is attributed to the additional slack matrices.

V. CONCLUSION

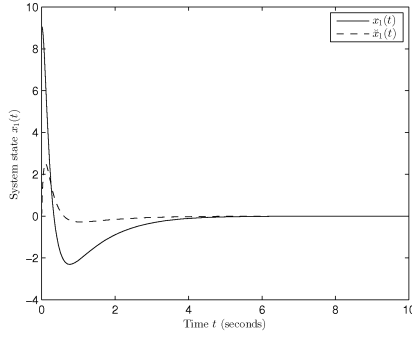
The stability of T-S FMB observer-control system has been investigated. Both the unmeasurable premise variables and membership-function-dependent approach have been considered to widen the applicability of the designed fuzzy observer-controller. Matrix decoupling technique has been employed to obtain convex stability conditions. The number of pre-defined scalars has been reduced by adequately choosing the augmented vector which is in favor of applying matrix decoupling technique. Simulation examples have been offered to demonstrate the relaxation of proposed observer-controller.

ACKNOWLEDGMENT

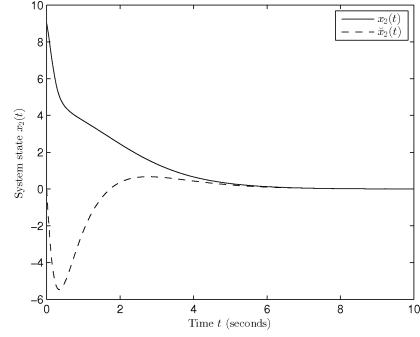
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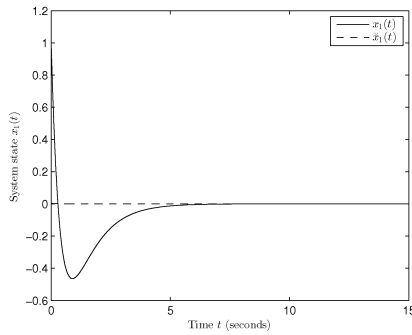


(a) System state $x_1(t)$ and estimated state $\hat{x}_1(t)$.

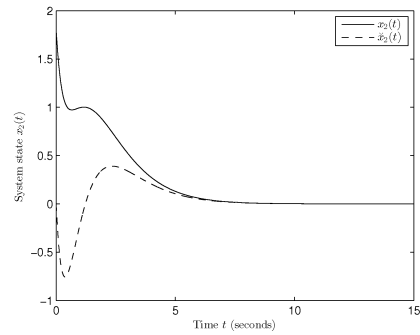


(b) System state $x_2(t)$ and estimated state $\hat{x}_2(t)$.

Fig. 1. Time responses of system states $x_1(t)$ and $x_2(t)$ and estimated states $\hat{x}_1(t)$ and $\hat{x}_2(t)$.



(a) System state $x_1(t)$ and estimated state $\hat{x}_1(t)$.



(b) System state $x_2(t)$ and estimated state $\hat{x}_2(t)$.

Fig. 2. Time responses of system states $x_1(t)$ and $x_2(t)$ and estimated states $\hat{x}_1(t)$ and $\hat{x}_2(t)$.

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